Level spacing of random matrices in an external source

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In an earlier work we considered a Gaussian ensemble of random matrices in the presence of a given external matrix source. The measure is no longer unitary invariant, and the usual techniques based on orthogonal polynomials, or on the Coulomb gas representation, are not available. Nevertheless the *n*-point correlation functions are still given in terms of the determinant of a kernel, known through an explicit integral representation. This kernel is no longer symmetric, however, and is not readily accessible to standard methods. In particular, finding the level spacing probability is always a delicate problem in Fredholm theory, and we have to reconsider the problem within our model. We find a class of universality for the level spacing distribution when the spectrum of the source is adjusted to produce a vanishing gap in the density of the state. The problem is solved through coupled nonlinear differential equations, which turn out to form a Hamiltonian system. As a result we find that the level spacing probability $p(s)$ behaves like $\exp[-Cs^{8/3}]$ for large spacing s; this is consistent with the asymptotic behavior $\exp[-Cs^{2\beta+2}]$, whenever the density of state behaves near the edge as $\rho(\lambda) \sim \lambda^\beta$. [S1063-651X(98)00112-3]

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I. INTRODUCTION

The level spacing distribution $p(s)$, first discussed by Wigner [1] for nuclear energy levels, has been studied extensively in random matrix theory [2], and the universality of $p(s)$ has been discussed for many cases, including the distribution of zeros in the Riemann ζ function [3]. The calculation of this level spacing distribution is always much more delicate than that of the *n*-point correlation functions. Indeed, when two neighboring levels are separated by some interval s, it implies that all the other eigenvalues are outside of this interval, and consequently it involves all the correlation functions of those eigenvalues. In the simplest Gaussian ensemble it took many years of development of the theory of Fredholm determinants, Dyson's inverse scattering approach, and τ functions before this problem was understood. For the Airy kernel which appears for the spacing in the vicinity of the edge of Wigner's semicircle, Tracy and Widom developed a technique through coupled nonlinear differential equations, which we have generalized here and applied to our previous work on random matrices in a matrix source [4].

The problem we had studied concerns a Hamiltonian $H = H_0 + V$, in which H_0 is an $N \times N$ nonrandom matrix with a known spectrum, but V is a random Gaussian potential. The probability measure for H, then being a Gaussian in $H - H_0$, is not unitary invariant. Consequently the standard approach, which consists of tracing out the unitary degrees of freedom, and then using orthogonal polynomials, or a Coulomb gas representation, is not available. However, we have obtained exact formulas, i.e., valid for matrices of finite size, for the *n*-level correlation functions, with the help of the

Itzykson-Zuber formula [5]. It is remarkable that these *n*-point functions are still given by a determinant of an $n \times n$ matrix whose elements are given by a kernel $K(\lambda, \mu)$, a well-known fact [3] when the orthogonal polynomials are available. However, in our problem, this kernel is no longer symmetric, but we know an exact integral representation for finite N.

If we had considered the level spacing distribution for those ensembles with a source, around a regular point of the spectrum, the usual proofs of universality would apply, and the end result, in the appropriate scaling limit, would be identical to the Wigner ensemble level spacing. However, near singularities of the spectrum, different universality classes appear. For instance, near the edge of the Wigner semicircle, in a region of size $N^{-2/3}$, a kernel expressed in terms of Airy functions controls the correlation functions; Tracy and Widom succeeded in finding the level spacing for this kernel.

In the source problem, we showed earlier that we can have gaps in the spectrum of H, when the randomness of V is not strong enough to bridge the gaps between the eigenstates of H_0. If we tune the randomness in order to reach the limiting situation in which a gap closes, a universal singularity appears at which the density of eigenvalues $\rho(\lambda)$ vanishes like $|\lambda|^{1/3}$. In this paper we obtain the level spacing distribution by an application of Fredholm theory from which we derive nonlinear differential equations, which remarkably form a Hamiltonian system. This generalizes earlier work, and makes it clear that the technique is general. For the Wigner case, with the sine kernel obtained by Gaudin [6],

$$K(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}, \tag{1.1}$$

Jimbo *et al.* [7] long ago obtained a closed equation for $E(s)$, the probability that the interval $(-s/2, s/2)$ is empty. Tracy and Widom [8] considered the Airy kernel

$$K(x,y) = \frac{A_i(x)A_i'(y) - A_i'(x)A_i(y)}{x - y} \quad (1.2)$$

[where $A_i(x)$ is an Airy function which satisfies $A_i''(x) = xA_i(x)$], and obtained $E(s)$ for the semi-infinite interval (s,∞), when s is near a singular edge of the semicircle. The method developed in this paper is easily applicable to those

two cases. In order to explain the method that we follow here, we present it for the sine kernel and for the Airy kernel in Appendixes A and B, following closely the derivation of Tracy and Widom [8–10].

In all those earlier cases, as well as for our source problem, the probability $E(s)$ of emptiness of the interval $(-s/2, s/2)$, is a Fredholm determinant

$$E(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_a^b \cdots \int_a^b \Pi_{k=1}^n dx_k \det[K(x_i, x_j)]_{i,j=1,\ldots,n} \quad (1.3)$$

if we choose $(a = -s/2, b = +s/2)$ for the interval (a,b). The sine and Airy kernels are symmetric: $K(x,y) = K(y,x)$. For the source problem, $K(x,y)$ is no longer symmetric under exchange of x and y, although its square under convolution yields back the kernel itself as for the simpler cases [4]. Starting with Fredholm theory for $E(s)$ and extending the analysis of Tracy and Widom [8–10], we derive a Hamiltonian system for our problem, leading to coupled nonlinear differential equations. From those equations one can determine the function $E(s)$, and in particular its asymptotic expansion for large s, which is interesting. For the sine kernel, it behaves as $E(s) \sim \exp[-(\pi^2/8)s^2]$; for the Airy kernel, it becomes $E(s) \sim \exp(-1/96s^3)$. In our previous work [4], we had applied a Padé analysis to a function $R(t)$ related to $E(s)$,

$$E(s) = \exp\left[\int_0^s R(\tilde{s})d\tilde{s}\right], \quad (1.4)$$

where the variable $\tilde{s}$ is defined as

$$\tilde{s} = \int_{-s/2}^{s/2} \rho(x)dx. \quad (1.5)$$

We had made the ansatz in our previous paper [4] that $R(\tilde{s})$ behaves like $\tilde{s}$ in the large-$\tilde{s}$ limit, namely, that $E(s)$ is

Gaussian in terms of $\tilde{s}$. More generally it has been conjectured, on the basis of a Coulomb gas representation [11,12], that for a density of state behaving as $\rho(x) \sim x^\beta$, then

$$E(s) \sim \exp[-Cs^{2\beta+2}]. \quad (1.6)$$

Indeed this ansatz agreed with the results for the sine kernel, for which $\beta = 0$, and for the Airy kernel, $\beta = \frac{1}{2}$. In our case, for which there is no Coulomb gas picture, the ansatz also agrees with our finding for the gap closing singularity, for which the density of state gives an exponent $\beta = \frac{1}{3}$. The result of the present analysis confirms this conjecture since it gives $E(s) \sim D\exp(-Cs^{8/3})$.

II. KERNEL AT THE CLOSURE OF A GAP

We consider an $N \times N$ Hamiltonian matrix $H = H_0 + V$, where H_0 is a given nonrandom Hermitian matrix, and V is a random Gaussian Hermitian matrix [13–17]. The probability distribution $P(H)$ is thus given by

$$P(H) = \frac{1}{Z}e^{-(N/2)\text{Tr }V^2} = \frac{1}{Z'}e^{-(N/2)\text{Tr}(H^2 - 2H_0 H)}. \quad (2.1)$$

For the n-point correlation functions, defined as

$$R_n(\lambda_1, \lambda_2, \ldots, \lambda_n) = \left\langle \frac{1}{N}\text{Tr }\delta(\lambda_1 - H)\frac{1}{N}\text{Tr }\delta(\lambda_2 - H) \cdots \frac{1}{N}\text{Tr }\delta(\lambda_n - H) \right\rangle, \quad (2.2)$$

we have earlier derived [16,17] the expression

$$R_2(\lambda_1, \lambda_2) = K_N(\lambda_1, \lambda_1)K_N(\lambda_2, \lambda_2) - K_N(\lambda_1, \lambda_2)K_N(\lambda_2, \lambda_1), \quad (2.3)$$

with the kernel

$$K_N(\lambda_1, \lambda_2) = (-1)^{N-1}\int \frac{dt}{2\pi} \oint \frac{du}{2\pi i} \prod_{\gamma=1}^N \left(\frac{a_\gamma - it}{u - a_\gamma}\right)\frac{1}{u - it}\exp\left(-\frac{N}{2}u^2 - \frac{N}{2}t^2 - Nit\lambda_1 + Nu\lambda_2\right), \quad (2.4)$$

in which the contour in the u plane encloses all the a_γ's. Similarly the n-point functions are given in terms of the determinant of the $n \times n$ matrix whose elements are given by this same kernel $K_N(\lambda_i, \lambda_j)$ [17,18]. In Ref. [15], this kernel $K_N(\lambda_1, \lambda_2)$ was considered in the large N limit, for fixed $N(\lambda_1 - \lambda_2)$. In this limit one can evaluate integral (2.4) by the saddle-point method. The result was found, up to a phase factor that we omit here, to be

$$K_N(\lambda_1,\lambda_2) = -\frac{1}{\pi y}\sin[\pi y\rho(\lambda_1)], \qquad (2.5)$$

where $y = N(\lambda_1 - \lambda_2)$. Apart from the scale dependence provided by the density of state ρ, the n-point correlation functions have thus a universal scaling limit, i.e., independent of the deterministic part H_0 of the random Hamiltonian.

In order to generate a tunable gap we consider the simple case for which the eigenvalues of H_0 are $\pm a$, each value being $N/2$ times degenerate. When the randomness of V is small, the average density of eigenvalues is made of two disjoint segments located around the points $\pm a$. When the randomness increases, one reaches a critical point at which the gap closes. When that happens in the vicinity of the origin of size $N^{-3/4}$ we find a class of universality for the density of states and for the n-point functions [4]. Then kernel (2.4) becomes

$$K_N(\lambda_1,\lambda_2) = (-1)^{(N/2)+1}\int\frac{dt}{2\pi}\oint\frac{du}{2\pi i}\left(\frac{a^2+t^2}{u^2-a^2}\right)^{N/2}\frac{1}{u-it}\exp\left(-\frac{N}{2}u^2-\frac{N}{2}t^2-Nit\lambda_1+Nu\lambda_2\right). \qquad (2.6)$$

From this expression, we obtain the density of state $\rho(\lambda) = K_N(\lambda,\lambda)$. The derivative of $\rho(\lambda)$ with respect to λ eliminates the factor $u-it$ in the denominator of Eq. (2.4), and leads to the factorized expression

$$\frac{1}{N}\frac{\partial}{\partial\lambda}\rho(\lambda) = -\phi(\lambda)\psi(\lambda), \qquad (2.7)$$

where

$$\phi(\lambda) = \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-(N/2)t^2+(N/2)\ln(a^2+t^2)-Nit\lambda}, \qquad (2.8)$$

$$\psi(\lambda) = \oint\frac{du}{2\pi i}e^{-(N/2)u^2-(N/2)\ln(a^2-u^2)+Nu\lambda}. \qquad (2.9)$$

For large N, we may apply to the two integrals defining the functions ϕ and ψ the saddle-point method. When λ_1 and λ_2 are near the origin the saddle points in the variables t and u move to the origin. Therefore, for obtaining the large N behavior of ϕ near $\lambda = 0$ we can expand the logarithmic term in powers of t. One sees readily that the coefficient of the quadratic term in t^2 vanishes for $a = 1$; in fact three saddle points are merging at the origin when a reaches 1. This is the critical value at which the gap closes. We must then expand in the exponential up to order t^4, and we obtain

$$\phi(\lambda) = \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{-(N/4)t^4-Nit\lambda}. \qquad (2.10)$$

Rescaling t to $N^{-1/4}t'$, and setting $\lambda = N^{-3/4}x$, we find that

$$\hat{\phi}(x) = N^{1/4}\phi(N^{-3/4}x) \qquad (2.11)$$

has a large N, finite x, limit given by

$$\hat{\phi}(x) = \frac{1}{\pi}\int_0^{\infty}dt\, e^{-(1/4)t^4}\cos(tx). \qquad (2.12)$$

It is immediate to verify that it satisfies the differential equation

$$\hat{\phi}'''(x) = x\hat{\phi}(x). \qquad (2.13)$$

From the integral representation (2.8) we obtain easily the Taylor expansion of this function at the origin

$$\hat{\phi}(x) = \frac{\sqrt{2}}{4\pi}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{1}{4}+\frac{m}{2}\right)(-1)^m 2^m x^{2m}}{(2m)!}, \qquad (2.14)$$

and its asymptotic behavior at large x,

$$\hat{\phi}(x) \sim \sqrt{\frac{2}{3\pi}}x^{-(1/3)}e^{-(3/8)x^{4/3}}\cos\left(\frac{3\sqrt{3}}{8}x^{4/3}-\frac{\pi}{6}\right). \qquad (2.15)$$

For the second function [Eq. (2.9)], in the scaling limit, N large, λ small, $N^{3/4}\lambda$ finite, we may expand up to order u^4, and define

$$\hat{\psi}(x) = N^{1/4}\psi(N^{-3/4}x). \qquad (2.16)$$

In the large N, finite x, limit, we find

$$\hat{\psi}(x) = \int_c\frac{du}{2\pi i}e^{(u^4/4)+ux}. \qquad (2.17)$$

The integral is over a path consisting of four lines of steepest descent in the complex u plane. Along these straight lines, the variable u is changed successively into $e^{\pm(\pi/4)i}u$ and $e^{\pm(3\pi/4)i}u$. This leads to

$$\hat{\psi}(x) = -\text{Im}\left[\frac{\omega}{\pi}\int_0^{\infty}e^{-u^4/4}(e^{xu\omega}-e^{-xu\omega})\right], \qquad (2.18)$$

in which $\omega = e^{\pi i/4}$. The function $\hat{\psi}(x)$ satisfies the differential equation

$$\hat{\psi}'''(x) = -x\hat{\psi}(x), \qquad (2.19)$$

and again, from Eq. (3.19) we find the Taylor expansion

$$\hat{\psi}(x) = -\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}(2n)!}{n!(4n+1)!} \qquad (2.20)$$

and the large x behavior

$$\hat{\psi}(x) \sim 2\sqrt{\frac{2}{3\pi}} x^{-1/3} e^{(3/8)x^{4/3}} \cos\left(\frac{3\sqrt{3}}{8} x^{4/3} + \frac{2\pi}{3}\right). \tag{2.21}$$

As shown in Ref. [4], one may express the whole kernel $K_N(\lambda_1,\lambda_2)$ of Eq. (3.1) in terms of the two functions $\hat{\phi}$ and $\hat{\psi}$ in the scaling limit. Defining

$$\lambda_1 = N^{-3/4} x, \quad \lambda_2 = N^{-3/4} y, \tag{2.22}$$

$$K(x,y) = N^{1/4} K_N(N^{3/4}\lambda_1, N^{3/4}\lambda_2). \tag{2.23}$$

in the large N, finite x and y, limit, we showed that

$$K(x,y) = \frac{\hat{\phi}'(x)\hat{\psi}'(y) - \hat{\phi}''(x)\hat{\psi}(y) - \hat{\phi}(x)\hat{\psi}''(y)}{x-y}. \tag{2.24}$$

Note that it follows from Eq. (2.12) that $\phi(x)$ is an even function, whereas $\psi(x)$ is odd [Eq. (2.18)]. This implies, in particular, that

$$K(-x,-y) = K(x,y). \tag{2.25}$$

Therefore, the density of state is given by

$$\hat{\rho}(x) = -[\hat{\phi}'(x)\hat{\psi}''(x) - \hat{\phi}''(x)\hat{\psi}'(x) + x\hat{\phi}(x)\hat{\psi}(x)]. \tag{2.26}$$

In the large x limit, it behaves as $\rho(x) \sim x^{1/3}$. Hereafter, we denote simply $\hat{\phi}$ and $\hat{\psi}$ by ϕ and ψ, respectively.

III. FREDHOLM THEORY

The level spacing function $E(s)$, the probability that there is no eigenvalue inside the interval $(-s/2,s/2)$ centered around the singular point $s=0$, is given by the Fredholm determinant

$$E(a,b) = \det[1-\hat{K}] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_a^b \cdots \int_a^b \Pi_{k=1}^n dx_k$$

$$\times \det[K(x_i,x_j)]_{i,j=1,\ldots,n} \tag{3.1}$$

if we choose for (a,b) the interval $(-s/2,s/2)$. The sine and Airy kernels are symmetric kernels, $K(x,y) = K(y,x)$, whereas our kernel in Eq. (2.24) is not symmetric, since the two functions $\phi(x)$ and $\psi(x)$ are different. Our notations here are as follows: the kernel $K(x,y)$ is defined by Eq. (2.24). However, it acts on the interval (a,b); therefore in Eq. (3.1) we have used the kernel $\hat{K}(x,y)$ defined by the restriction of K to the interval:

$$\hat{K}(x,y) = K(x,y)\theta(y-a)\theta(b-y) = K(x,y)\Theta(y), \tag{3.2}$$

in which $\theta(x)$ is the Heaviside function, and for convenience we have used the notation

$$\Theta(y) = \theta(y-a)\theta(b-y). \tag{3.3}$$

In order to calculate the derivative of the logarithm of $E(a,b)$ with respect to the end points one writes

$$\ln E(a,b) = \text{Tr} \ln(1-\hat{K}), \tag{3.4}$$

and thus

$$\frac{\partial \ln E(a,b)}{\partial b} = -\text{Tr} \frac{1}{1-\hat{K}} \frac{\partial \hat{K}}{\partial b}. \tag{3.5}$$

From Eq. (3.2), we find

$$\frac{\partial \hat{K}(x,y)}{\partial b} = K(x,b)\delta(y-b), \tag{3.6}$$

and therefore, if we introduce the Fredholm resolvent $\tilde{K}(b,b)$, defined by

$$\tilde{K} = \frac{\hat{K}}{1-\hat{K}}, \tag{3.7}$$

we obtain

$$\frac{\partial \ln E(a,b)}{\partial b} = -\tilde{K}(b,b). \tag{3.8}$$

Similarly the derivative with respect to a is

$$\frac{\partial \ln E(a,b)}{\partial a} = \tilde{K}(a,a). \tag{3.9}$$

Therefore, when we choose for (a,b) the interval $(-s/2,s/2)$, we have

$$\frac{d\ln E(s)}{ds} = \frac{1}{2}\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial a}\right)\ln E(s)|_{b=-a=s/2} = -\tilde{K}\left(\frac{s}{2},\frac{s}{2}\right). \tag{3.10}$$

This leads to

$$E(s) = \exp\left[-\int_0^s \tilde{K}\left(\frac{s'}{2},\frac{s'}{2}\right)ds'\right]. \tag{3.11}$$

We now define six functions, obtained by acting with the operator $(1-\hat{K})^{-1}$ on the functions $\phi(x),\psi(x)$ and their first two derivatives,

$$q_0(x) = (1-\hat{K})^{-1}\phi(x) = \phi(x) + \int_a^b K(x,y)\phi(y)dy$$

$$+ \int_a^b \int_a^b K(x,y)K(y,z)\phi(z)dy\,dz + \cdots. \tag{3.12}$$

We find it easier to use Dirac's notations

$$q_0(b,a;x) = \langle x|\frac{1}{1-\hat{K}}|\phi\rangle. \tag{3.13}$$

Similarly we define

$$q_n(b,a;x) = \langle x|\frac{1}{1-\hat{K}}|\phi^{(n)}\rangle, \tag{3.14}$$

where $\phi^{(n)}(x)$ is the nth derivative of $\phi(x)$. For the function $\psi(x)$, we act with the operator $(1-\hat{K})^{-1}$ on bras, i.e., dual vectors, rather than on kets. Because of the lack of symmetry of the kernel K this introduces a kernel $\hat{L}$, and

$$p_n(b,a;x)=(-1)^{n-1}\langle\psi^{(2-n)}|\frac{1}{1-\hat{L}}|x\rangle, \qquad (3.15)$$

where

$$\hat{L}(y,x)=\Theta(y)K(y,x). \qquad (3.16)$$

When we set $x=b$, and $a=-b$, then the six functions q_n and p_n become functions of the single variable b, which we denote as

$$Q_n(b)=q_n(b,-b;b), \quad P_n(b)=p_n(b,-b;b). \qquad (3.17)$$

The calculation of the derivatives of these functions $Q_n(b)$ and $P_n(b)$ implies considering separate variations in the functions $q_n(b,a;x)$ and $p_n(b,a;x)$ with respect to b, a, and x, before we set $a=-b$ and $x=b$. The calculation is tedious, but not difficult, and we give more details in Appendix C. The resulting differential equations are

$$\dot{Q}_0=Q_1+\frac{2}{b}Q_1P_1Q_0,$$

$$\dot{Q}_1=Q_2-\frac{2}{b}Q_1^2P_1-Q_0u,$$

$$\dot{Q}_2=bQ_0+\frac{2}{b}Q_1P_1Q_2-Q_1v,$$

$$\dot{P}_0=-bP_2-\frac{2}{b}Q_1P_1P_0+P_1u,$$

$$\dot{P}_1=-P_0+\frac{2}{b}Q_1P_1^2+P_2v,$$

$$\dot{P}_2=-P_1-\frac{2}{b}Q_1P_1P_2, \qquad (3.18)$$

where a dot means taking the derivative with respect to b. The two auxiliary functions u and v are defined as

$$u=\langle\psi|q_1\rangle, \quad v=\langle\psi'|q_0\rangle, \qquad (3.19)$$

and they satisfy

$$\dot{u}=-2P_2Q_1, \quad \dot{v}=2P_1Q_0. \qquad (3.20)$$

From these equations, we successively find the relations

$$u+v=-2P_2Q_0, \qquad (3.21)$$

then

$$\dot{Q}_0=Q_1\left(1+\frac{\dot{v}}{b}\right), \quad \dot{P}_2=-P_1\left(1-\frac{\dot{u}}{b}\right). \qquad (3.22)$$

Using Eqs. (3.20) and (3.21), we obtain, from the second equation of Eq. (3.18),

$$-2P_2Q_2=\ddot{u}-\frac{\dot{u}\dot{v}}{u+v}+\frac{2}{b}\frac{\dot{v}\dot{u}^2}{u+v}+u(u+v), \qquad (3.23)$$

and, using the fifth equation of Eq. (3.18), we have

$$-2Q_0P_0=\ddot{v}-\frac{\dot{u}\dot{v}}{u+v}-\frac{2}{b}\frac{\dot{u}\dot{v}^2}{u+v}+v(u+v). \qquad (3.24)$$

Taking the derivatives of these two equations and using the third and sixth equations of Eq. (3.18), we end up with two coupled nonlinear equations:

$$\frac{d^3u}{db^3}+\left(\frac{2\dot{u}}{b}-1\right)\left[b(u+v)+\frac{1}{u+v}(\ddot{v}\dot{u}+2\dot{v}\ddot{u})\right.$$
$$\left.-\frac{\dot{u}\dot{v}}{(u+v)^2}(2\dot{v}+\dot{u})\right]-\frac{2\dot{v}(\dot{u})^2}{b^2(u+v)}=0, \qquad (3.25)$$

$$\frac{d^3v}{db^3}+\left(\frac{2\dot{v}}{b}+1\right)\left[b(u+v)-\frac{1}{u+v}(\ddot{u}\dot{v}+2\dot{u}\ddot{v})\right.$$
$$\left.+\frac{\dot{u}\dot{v}}{(u+v)^2}(2\dot{u}+\dot{v})\right]+\frac{2\dot{u}(\dot{v})^2}{b^2(u+v)}=0. \qquad (3.26)$$

In the large b limit, the asymptotic behavior of the solutions of these equations is obtained under the forms

$$u=\frac{b^2}{4}+\frac{A}{2}b^{2/3}+\cdots, \qquad (3.27)$$

$$v=-\frac{b^2}{4}+\frac{A}{2}b^{2/3}+\cdots. \qquad (3.28)$$

Inserting these expressions in Eq. (3.25), we find A from the coefficient of the terms of order $b^{1/3}$,

$$A=-(\tfrac{1}{4})^{1/3}. \qquad (3.29)$$

The kernel $\tilde{K}(b,b)$ may then be expressed as

$$\tilde{K}(b,b)=bP_2Q_0+Q_2P_1+Q_1P_0-uP_1Q_0-vP_2Q_1$$
$$-\frac{1}{2b}(P_1^2Q_1^2-Q_2P_2-Q_0P_0)^2. \qquad (3.30)$$

Remarkably this kernel at coinciding points is in fact a Hamiltonian, that we denote as

$$H(b)=\tilde{K}(b,b), \qquad (3.31)$$

from which Hamilton's equations

$$\dot{Q}_n=\frac{\partial H}{\partial P_n}, \quad \dot{P}_n=-\frac{\partial H}{\partial Q_n} \qquad (3.32)$$

coincide with the differential equations (3.18). This allows one to obtain a simple expression, for its derivative with respect to b becomes

$$\frac{dH(b)}{db} = \frac{\partial H}{\partial b} = Q_0 P_2 + \frac{2}{b^2} P_1^2 Q_1^2. \qquad (3.33)$$

In terms of u and v, it becomes

$$\frac{dH(b)}{db} = -\frac{u+v}{2} + \frac{(\dot{u}\dot{v})^2}{2b^2(u+v)^2}. \qquad (3.34)$$

Thus, from the previous result, we obtain the large b behavior

$$\frac{dH(b)}{db} \sim 5 \times 2^{-11/3} b^{2/3}. \qquad (3.35)$$

Integrating once, for the Hamiltonian we find $H(b) \sim 3 \times 2^{-11/3} b^{5/3} \sim 3 \times 2^{-16/3} s^{5/3}$ in the large s limit.

From Eq. (3.11), we have

$$E(s) = D \exp\left[-\int_0^s H(s') ds'\right] \sim \exp[-9 \times 2^{-25/3} s^{8/3}], \qquad (3.36)$$

in which D is an undetermined constant.

We have thus derived the exponent $s^{8/3}$ which was a mere conjecture in our previous paper [4]. There we had performed a simple Padé analysis of the small s expansion, and assumed that it was Gaussian at large s in the variable $\tilde{s}$. This led us to the estimate

$$E(s) \sim D \exp[-0.332 \tilde{s}^2], \qquad (3.37)$$

or, since $\tilde{s} \sim (3\sqrt{3}/2\pi)(\frac{1}{2})^{4/3} s^{4/3}$, $E(s) \sim D \exp[-0.0358 s^{8/3}]$. Our analytic result [Eq. (3.22)] gives $E(s) \sim D \exp[-0.0280 s^{8/3}]$. Thus the estimation by a simple Padé analysis was not too far from the exact result.

IV. SUMMARY AND DISCUSSION

In this paper, we have investigated the level spacing probability for the case in which two edge singularities collapse. By use of the Fredholm theory, we have derived an expression for the level spacing probability, whose logarithmic derivative turns out to act as a Hamiltonian. The same strategy also allows one to solve simpler cases, such as the sine-kernel (relevant to the level spacing for ordinary nonsingular) points of the spectrum and the Airy kernel (which applies to a single edge singularity). The corresponding Hamilton's equations fully determine the level spacing, and in particular one can analytically obtain its asymptotic expansion at large spacing s. This allowed us to confirm the conjecture that we had made in our previous paper, on the asymptotic Gaussian behavior of the level spacing in terms of the variable $\tilde{s} = \int_{-s/2}^{s/2} \rho(x) dx$, which is the number of eigenvalues in the interval of size s [4]. We have thus derived here that the level spacing probability $E(s)$ behaves like $\exp[-Cs^{8/3}]$, with a constant C that we have analytically determined. More generally the three cases that we have solved are consistent with the asymptotic Gaussian behavior of $E(s)$ with respect to $\tilde{s}$, i.e., to $E(s) \sim \exp[-Cs^{2\beta+2}]$ for large spacing s, whenever the interval of size s is around a point at which the density of state behaves as $\rho(\lambda) \sim \lambda^\beta$. The scaling variable s is in fact related to the actual interval between the eigenvalues by absorbing a power $N^{1/(\beta+1)}$. If we let N go to infinity first, at fixed interval, in which case s is also large, a finite limit of $[\ln E(s)/N^2]$, for large N, implies that $E(s)$ does fall for large s as $E(s) \sim \exp[-Cs^{2\beta+2}]$. In other words, $E(s)$ behaves for large N, at a fixed interval, as a partition function.

APPENDIX A: LEVEL SPACING PROBABILITY FOR THE SINE KERNEL

The sine kernel is defined by

$$K(x,y) = \frac{\phi(x)\phi'(y) - \phi(y)\phi'(x)}{x-y}, \qquad (A1)$$

where $\phi(x) = \sin x$ satifies $\phi''(x) = -\phi(x)$. (For convenience, in the normalization we have absorbed the usual factor π). We consider $q(x)$ and $p(x)$ defined by

$$q(x) = \langle x| \frac{1}{1-\hat{K}} |\phi\rangle, \quad p(x) = \langle \phi'| \frac{1}{1-\hat{K}} |x\rangle, \quad (A2)$$

and the Fredholm resolvent $\tilde{K}$

$$\tilde{K} = \frac{\hat{K}}{1-\hat{K}}. \qquad (A3)$$

We have

$$(x-y)\tilde{K} = \langle x|[X,\tilde{K}]|y\rangle = \langle x|\left[X, \frac{1}{1-\hat{K}}\right]|y\rangle$$

$$= \langle x| \frac{1}{1-\hat{K}} [X,\hat{K}] \frac{1}{1-\hat{K}} |y\rangle. \qquad (A4)$$

The definition (A1) of the kernel reads

$$[X,K] = |\phi\rangle\langle\phi'| - |\phi'\rangle\langle\phi| \qquad (A5)$$

since

$$(x-y)K(x,y) = \langle x|[X,K]|y\rangle. \qquad (A6)$$

Thus, from Eq. (A4), we obtain

$$\tilde{K}(x,y) = \frac{q(x)p(y) - q(y)p(x)}{x-y}. \qquad (A7)$$

Since the functions $q(x)$ and $p(x)$ depend upon the interval (a,b), we denote them more precisely as $q(b,a;x)$ and $p(b,a;x)$. We then set $x=b$ and vary b, i.e., take the derivative of $q(b,a;b)$ at fixed a:

$$\frac{\partial q(b,a;b)}{\partial b}=\langle b|D\frac{1}{1-\hat{K}}|\phi\rangle+\langle b|\frac{1}{1-\hat{K}}\frac{\partial\hat{K}}{\partial b}\frac{1}{1-\hat{K}}|\phi\rangle,$$

$$(A8)$$

where D is the derivative operator: $\langle x|D|f\rangle=f'(x)$. From the definition of $\hat{K}$, we have

$$\frac{\partial\hat{K}(x,y)}{\partial b}=K(x,y)\delta(b-y)=K|b\rangle\langle b|.\qquad(A9)$$

Thus the second term of Eq. (A8) becomes $\tilde{K}(b,b)q(b)$. The first term of Eq. (A8) becomes

$$\langle b|D\frac{1}{1-\hat{K}}|\phi\rangle=p(b)+\langle b|\left[D,\frac{1}{1-\hat{K}}\right]|\phi\rangle$$

$$=p(b)+\langle b|\frac{1}{1-\hat{K}}[D,\hat{K}]\frac{1}{1-\hat{K}}|\phi\rangle.$$

$$(A10)$$

Since

$$\langle x|[D,\hat{K}]|y\rangle=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\hat{K}(x,y)$$

$$=K(x,y)[\delta(y-a)-\delta(y-b)],\quad(A11)$$

we obtain

$$[D,\hat{K}]=K|a\rangle\langle a|-K|b\rangle\langle b|.\qquad(A12)$$

From Eqs. (A8) and (A10), we have

$$\frac{\partial q(b,a;b)}{\partial b}=p(b,a;b)+\tilde{K}(b,a)q(b,a;a).\quad(A13)$$

Similarly, we have, for $p(b)$,

$$\frac{\partial p(b,a;b)}{\partial b}=-q(b,a;b)+\tilde{K}(b,a)p(b,a;a).\quad(A14)$$

We denote $q(b,-b;b)$ by $Q(b)$,

$$\frac{dQ(b)}{db}=\left.\frac{\partial q(b,a;b)}{\partial b}\right|_{a=-b}-\left.\frac{\partial q(b,a;b)}{\partial a}\right|_{a=-b}.$$

$$(A15)$$

Since

$$\frac{\partial q(b,a;b)}{\partial a}=\langle b|\frac{1}{1-\hat{K}}\left(\frac{\partial\hat{K}}{\partial a}\right)\frac{1}{1-\hat{K}}|\phi\rangle=-\tilde{K}(b,a)q(b,a;a),$$

$$(A16)$$

we have

$$\dot{Q}(b)=P(b)+2\tilde{K}(b,-b)Q(-b),\qquad(A17)$$

where

$$\tilde{K}(b,-b)=\frac{Q(b)P(b)}{b}.\qquad(A18)$$

Note that $Q(-b)=-Q(b)$ and $P(-b)=P(b)$. Similarly we have an equation for $P(b)$. Thus we obtain finally

$$\dot{Q}=P\left(1-\frac{2Q^2}{b}\right),\quad\dot{P}=Q\left(\frac{2P^2}{b}-1\right).\qquad(A19)$$

The function $\tilde{K}(b,b)$ is related to P and Q as

$$\tilde{K}(b,b)=P^2+Q^2-\frac{2P^2Q^2}{b},\qquad(A20)$$

which gives the logartihmic derivative of the level spacing probability $E(s)$. Noting $b=s/2$, we have

$$\frac{dQ}{ds}=\frac{P}{2}\left(1-\frac{4}{s}Q^2\right),\quad\frac{dP}{ds}=\frac{Q}{2}\left(\frac{4P^2}{s}-1\right).\quad(A21)$$

In the large s limit, we have $Q\sim P\sim s^{1/2}/2$. From Eq. (A20), we obtain $H(s)=\tilde{K}(b,b)\sim s/4$. The level spacing probability $E(s)$ thus behaves, in the large s limit, as

$$E(s)\sim\exp\left[-\int_0^{\pi s}\frac{s'}{4}ds'\right]\sim\exp\left[-\frac{\pi^2}{8}s^2\right].\quad(A22)$$

For small s, we have, by solving Eq. (A21) iteratively, with the initial conditions $P(0)=1$ and $Q(0)=0$,

$$Q=\frac{s}{2}-\frac{s^3}{48}+O(s^5),\quad P=1+s+\frac{7}{8}s^2+O(s^3).$$

$$(A23)$$

Then, from Eq. (A20), $H(b)=\tilde{K}(b,b)$ behaves for small s as

$$H(s)=1+s+s^2+O(s^3).\qquad(A24)$$

This leads to

$$E(s)=\exp\left[-\int_0^s H(s')ds'\right]=1-s+O(s^4),\quad(A25)$$

which is consistent with all the well-known results on this well-studied case [3].

APPENDIX B: LEVEL SPACING PROBABILITY FOR THE AIRY KERNEL

The sine kernel applies to a regular point of the spectrum. However when one studies the vicinity of the edge of the spectrum, in the appropriate scaling limit, the correlation functions and the level spacing are given by an Airy kernel. The level spacing was studied by Tracy and Widom [8]. Here we repeat the same technique. Consider the interval $(-s/2,s/2)$ as in the sine case. We denote the Airy function $A_i(x)$ by $\phi(x)$; it satisfies $\phi''(x)=x\phi(x)$. We use the same notation $q(x)$ and $p(x)$ as in Eq. (A2). As for the sine case, the Fredholm resolvent $\tilde{K}(a,b)$ is given by

$$\tilde{K}(a,b)=\frac{q(a)p(b)-p(a)q(b)}{a-b}.\qquad(B1)$$

We have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)K(x,y)=-\phi(x)\phi(y). \tag{B2}$$

Therefore, we have

$$[D,K]=-|\phi\rangle\langle\phi|\Theta+K|a\rangle\langle a|-K|b\rangle\langle b|. \tag{B3}$$

Thus, by the same procedure as in the sine case, we obtain

$$\frac{\partial q(b)}{\partial b}=p(b)-qu+\widetilde{K}(b,a)q(a), \tag{B4}$$

$$\frac{\partial p}{\partial b}=bq(b)+up(b)-2q(b)v+\widetilde{K}(b,a)p(a), \tag{B5}$$

where

$$u=\langle\phi|q\rangle,\quad v=\langle\phi|p\rangle. \tag{B6}$$

The Fredholm resolvent $\widetilde{K}(b,b)$ becomes

$$\widetilde{K}(b,b)=p^2(b)-bq^2(b)-2up(b)q(b)+2q^2(b)v$$
$$+\frac{1}{b-a}[q(b)p(a)-p(b)q(a)]$$
$$\times[q(a)p(b)-p(a)q(b)]. \tag{B7}$$

Again $H(b)=\widetilde{K}(b,b)$ acts as a Hamiltonian, since we have

$$\frac{\partial H(b)}{\partial p(b)}=2\frac{\partial q(b)}{\partial b},\quad \frac{\partial H(b)}{\partial q(b)}=-2\frac{\partial p(b)}{\partial b}. \tag{B8}$$

The derivative of the Hamiltonian becomes

$$\frac{dH(b)}{db}=-q^2(b)+\frac{[q(b)p(a)-p(b)q(a)]^2}{(b-a)^2}. \tag{B9}$$

In the Airy kernel, due to the parity around the edge, $H(a)$ and $H(b)$ are different. The quantities $q(b)$ and $p(b)$ become exponentially small in the large b limit ($b>0$), the same as in the Airy function. We have, for $a=-s/2$,

$$\frac{\partial q(a)}{\partial a}=p(a)-q(a)u-\widetilde{K}(a,b)q(b), \tag{B10}$$

$$\frac{\partial p(a)}{\partial a}=aq(a)+up(a)-2q(a)v-\widetilde{K}(a,b)p(b). \tag{B11}$$

The Hamiltonian $H(a)$ becomes

$$H(a)=p^2(a)-aq^2(a)-2up(a)q(a)+2q^2(a)v$$
$$+\widetilde{K}(a,b)[q(a)p(b)-p(a)q(b)]. \tag{B12}$$

Since $\widetilde{K}(a,b)$ can be neglected for the large b limit, and we have a relation $u^2-2v=q^2$, we obtain

$$\frac{d^2q(a)}{da^2}=aq(a)+2q^3(a). \tag{B13}$$

We are looking for a solution which behaves as a power of s for large s, and thus for which $d^2q(a)/da^2$ can be neglected asymptotically. This leads to $q(a)\sim\pm\sqrt{s}/2$ for large s, and thus to $H(a)\sim s^2/16$ in the large s limit. Then we obtain $E(s)\sim\exp[-s^3/96]$. In order to check the usefulness of the Padé analysis, we now expand at small s.

The Airy function $A(x)$ has the Taylor expansion for small x,

$$A(x)=c_1\left[1+\frac{1}{6}x^3+\frac{4}{6!}x^6+\cdots\right]$$
$$-c_2\left[x+\frac{2}{4!}x^4+\frac{2\times5}{7!}x^7+\cdots\right], \tag{B14}$$

where $c_1=3^{-2/3}/\Gamma(2/3)$ and $c_2=3^{-1/3}/\Gamma(1/3)$. If we use $\widetilde{s}$, defined in Eq. (1.5), it is related to s in this Airy case by

$$\widetilde{s}=c_2^2s+\frac{c_1c_2}{12}s^3-\frac{c_1^2}{960}s^5-\frac{c_2^2}{16\,128}s^7+O(s^9). \tag{B15}$$

The level spacing $E(s)$ is thus expanded in powers of $\widetilde{s}$ as

$$E(s)=1-\widetilde{s}+\frac{1}{6}\left(\frac{c_1c_2^3}{3}-\frac{c_1^4}{8}\right)\frac{1}{c_2^8}\widetilde{s}^4+\cdots=1-\widetilde{s}+0.544\,868\widetilde{s}^4$$
$$-42.5418\widetilde{s}^6+O(\widetilde{s}^8). \tag{B16}$$

We now apply a Padé analysis to $H(s)$, or rather to

$$R(\widetilde{s})=\frac{d}{d\widetilde{s}}\ln E(\widetilde{s})=-\frac{1+a_1\widetilde{s}+a_2\widetilde{s}^2+a_3\widetilde{s}^3}{1+b_1\widetilde{s}+b_2\widetilde{s}}, \tag{B17}$$

where we have $a_1=754.156$, $a_2=1640.76$, $a_3=1638.58$, $b_1=753.156$, and $b_2=886.601$. From $E(\widetilde{s})$, differentiating twice with respect to $\widetilde{s}$, we obtain $p(\widetilde{s})$, which is slightly different from the usual ''Wigner surmise'' function of the sine case.

APPENDIX C: LEVEL SPACING PROBABILITY FOR THE GAP CLOSURE KERNEL

We now consider the kernel

$$K(x,y)=\frac{\phi'(x)\psi'(y)-\phi''(x)\psi(y)-\phi(x)\psi''(y)}{x-y}, \tag{C1}$$

or, in operator notations,

$$[X,K]=|\phi'\rangle\langle\psi'|-|\phi''\rangle\langle\psi|-|\phi\rangle\langle\psi''|. \tag{C2}$$

Similarly to Eq. (A4), we have

$$(x-y)\widetilde{K}=\langle x|\frac{1}{1-\hat{K}}[X,\hat{K}]\frac{1}{1-\hat{K}}|y\rangle=q_1(x)p_1(y)$$
$$-q_2(x)p_0(y)-q_0(x)p_2(y). \tag{C3}$$

The derivative of $q_n(b,a;b)$ for a fixed a becomes

$$\frac{\partial q_n(b,a;b)}{\partial b} = \langle b|D\frac{1}{1-\hat{K}}|\phi^{(n)}\rangle + \langle b|\frac{1}{1-\hat{K}}\left(\frac{\partial\hat{K}}{\partial b}\right)\frac{1}{1-\hat{K}}|\phi^{(n)}\rangle = q_{n+1}(b,a;b) + \langle b|\frac{1}{1-\hat{K}}[D,\hat{K}]\frac{1}{1-\hat{K}}|\phi^{(n)}\rangle$$

$$+ \langle b|\frac{\hat{K}}{1-\hat{K}}|b\rangle\langle b|\frac{1}{1-\hat{K}}|\phi^{(n)}\rangle. \quad (C4)$$

We also have

$$\langle x|[D,\hat{K}]|y\rangle = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)K(x,y) + \langle x|K|a\rangle\langle a|y\rangle - \langle x|K|b\rangle\langle b|y\rangle. \quad (C5)$$

The first term is simply $-\phi(x)\psi(y)$. This leads to

$$[D,\hat{K}] = -|\phi\rangle\langle\psi|\Theta + K|a\rangle\langle a| - K|b\rangle\langle b|. \quad (C6)$$

Therefore,

$$\frac{\partial q_n(b,a;b)}{\partial b} = q_{n+1} + \tilde{K}(b,a)q_n(a) - q_0(b)\langle\psi|q_n\rangle \quad (C7)$$

and

$$q_3 = \langle x|\frac{1}{1-\hat{K}}X|\phi\rangle = \langle x|X\frac{1}{1-\hat{K}}|\phi\rangle + \langle x|\left[\frac{1}{1-\hat{K}},X\right]|\phi\rangle$$

$$= xq_0(x) + \langle x|\frac{1}{1-\hat{K}}[\hat{K},X]\frac{1}{1-\hat{K}}|\phi\rangle$$

$$= xq_0 - v_2 q_1 + u_1 q_2 + v_3 q_0, \quad (C8)$$

where $u_1 = \langle\psi|q_0\rangle$, $v_2 = \langle\psi'|q_0\rangle$, and $v_3 = \langle\psi''|q_0\rangle$.

The function $p_n(x)$ is defined by

$$p_n(x) = (-1)^{n-1}\langle\psi^{(2-n)}|\frac{1}{1-\hat{L}}|x\rangle, \quad (C9)$$

where

$$\hat{L}(y,x) = \Theta(y)K(y,x). \quad (C10)$$

We have

$$[D,\hat{L}] = -\Theta|\phi\rangle\langle\psi| + |a\rangle\langle a|K - |b\rangle\langle b|K, \quad (C11)$$

in which Θ is a local operator defined by

$$\langle y|\Theta|y'\rangle = \delta(y-y')\theta(y-a)\theta(b-y). \quad (C12)$$

Thus we obtain

$$\frac{\partial p_n(b)}{\partial b} = -p_{n-1}(b) - p_0(b)\langle\psi^{(2-n)}|q_0\rangle + p_n(a)\tilde{K}(a,b), \quad (C13)$$

with $p_{-1}(x)$ obtained as

$$p_{-1}(x) = -\langle\psi'''|\frac{1}{1-\hat{L}}|x\rangle$$

$$= xp_2(x) - \langle\psi|\frac{1}{1-\hat{L}}[x,\hat{L}]\frac{1}{1-\hat{L}}|x\rangle$$

$$= -xp_0(x) - p_1(x)\langle\psi|q_1\rangle$$

$$- p_2\langle\psi|q_2\rangle - p_0(x)\langle\psi|q_0\rangle, \quad (C14)$$

where

$$\langle y|[X,\hat{L}]|x\rangle = (y-x)\Theta(y)K(y,x)$$

$$= \Theta(y)(|\phi'\rangle\langle\psi'| - |\phi''\rangle\langle\psi| - |\phi\rangle\langle\psi''|). \quad (C15)$$

The function $\phi(x)$ is an even function of x, and $q_0(x)$ becomes an even function. The function $\psi(x)$ is an odd function. Therefore, we have $u_1 = v_3 = 0$ for the interval $(-b,b)$. We also have $\langle\psi|q_2\rangle = \langle\psi|q_0\rangle = \langle\psi''|q_0\rangle = 0$. Nonvanishing quantities are $u_2 = \langle\psi|q_1\rangle$, and $v_2 = \langle\psi'|q_0\rangle$. We denote them simply by u and v.

Noting that

$$\dot{Q}_n = \frac{\partial q_n}{\partial b}\Big|_{a=-b} - \frac{\partial q_n}{\partial b}\Big|_{a=-b}, \quad (C16)$$

we obtain Eq. (3.18). The equations for $\dot{P}_n$ in Eq. (3.18) are also obtained similarly.

The derivative of $u = u_2$ becomes

$$\dot{u} = -2P_2(b)Q_1(b). \quad (C17)$$

This is obtained as

$$\frac{\partial u}{\partial b} = \langle\psi|\Theta\frac{1}{1-\hat{K}}\frac{\partial\hat{K}}{\partial b}\frac{1}{1-\hat{K}}|\phi'\rangle + \psi(b)q_1(b). \quad (C18)$$

Using

$$\frac{\partial\hat{K}}{\partial b} = K(x,y)\delta(y-b) = K|b\rangle\langle b|, \quad (C19)$$

we have

$$\frac{\partial u}{\partial b} = -p_2(b)q_1(b). \quad (C20)$$

The derivative of u by a becomes, similarly,

$$\frac{\partial u}{\partial a} = p_2(a) q_1(a). \tag{C21}$$

Putting $a = -b$, we have

$$\dot{u} = \left.\frac{\partial u}{\partial b}\right|_{a=-b} - \left.\frac{\partial u}{\partial a}\right|_{a=-b} = -2 P_2(b) Q_1(b). \tag{C22}$$

This is Eq. (3.19).

APPENDIX D: MODIFIED KERNEL

We have considered the case of the fixed external source eigenvalues at $a = \pm 1$. Here, we take this external eigenvalue a as $a^2 = 1 + 2N^{-1/2}\alpha$. The parameter α measures the approach to the limit $a = \pm 1$ in the large N limit, where the gap is closed. This change of a modifies the function $\phi(x)$ and $\psi(x)$. We have, in the large scaling limit,

$$\phi(\lambda) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-(t^4/4) - \alpha t^2 + it\lambda}, \tag{D1}$$

and it satisfies

$$\phi''' - 2\alpha\phi' - \lambda\phi = 0. \tag{D2}$$

The equation, which $\psi(\lambda)$ satisfies, is also modified as

$$\psi''' - 2\alpha\psi' + \lambda\psi = 0. \tag{D3}$$

Following the same procedure in the Appendix B in Ref. [4], we have a kernel, which is slightly different from (3.14):

$$K(x,y) = \frac{\phi'(x)\psi'(y) - \phi''(x)\psi(y) - \phi(x)\psi''(y) + 2\alpha\phi(x)\psi(y)}{x - y}. \tag{D4}$$

However, the density of state $\rho(x)$ is expressed by the same equation as Eq. (2.26),

$$\rho(x) = -[\phi'(x)\psi''(x) - \phi''(x)\psi'(x) + x\phi(x)\psi(x)]. \tag{D5}$$

The large λ behavior of $\phi(\lambda)$, for a fixed α, is obtained by a saddle-point method. If we make a change of t by $\lambda^{1/3}t$, we find that the new term αt^2 becomes negligible compared with other terms, which becomes order of $\lambda^{4/3}$. Then we obtain the large x behavior of $\rho(x)$ as $x^{1/3}$ same as before.

Using the same definitions for q_n and p_n, we have an equation

$$\tilde{K}(a,b) = \frac{q_1(a)p_1(b) + q_0(a)p_0(b) + q_2(a)p_2(b) - 2\alpha q_0(a)p_2(b)}{a - b}. \tag{D6}$$

Although there are modifications in the differential equations for $q(b)$ and $p(b)$, the derivative of the Hamiltonian is given by Eq. (3.33), which gives the same asymptotic behavior of $E(s)$. We note that the function $\phi(x)$ in Eq. (D1) appears for the second Painlevé A_2 Garnier system (Appendix A in Ref. [4]), and the function $\phi(x)$ satisfies the coupled linear partial differential equations about x and α.

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[1] E. P. Wigner, Proc. Cambridge Philos. Soc. **47**, 790 (1951).
[2] F. J. Dyson, J. Math. Phys. **13**, 90 (1972).
[3] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, New York, 1991).
[4] E. Brézin, S. Hikami, Phys. Rev. E **57**, 4140 (1998).
[5] C. Itzykson and J.-B. Zuber, J. Math. Phys. **21**, 411 (1980).
[6] M. Gaudin, Nucl. Phys. **25**, 447 (1961).
[7] M. Jimbo, T. Miwa, Y. Mori, and M. Sato, Physica D **1**, 80 (1980).
[8] C. A. Tracy and H. Widom, Commun. Math Phys. **159**, 151 (1994).
[9] C. A. Tracy and H. Widom, Commun. Math Phys. **161**, 289 (1994).
[10] C. A. Tracy and H. Widom, Commun. Math Phys. **163**, 33 (1994).
[11] P. J. Forrester, Nucl. Phys. B **402**, 709 (1993).
[12] Y. Chen, K. Eriksen, and C. A. Tracy, J. Phys. A **28**, L207 (1995).
[13] E. Brézin, S. Hikami, and A. Zee, Phys. Rev. E **51**, 5442 (1995).
[14] E. Brézin, S. Hikami, and A. Zee, Nucl. Phys. B **464**, 411 (1996).
[15] E. Brézin and S. Hikami, Nucl. Phys. B **479**, 697 (1996).
[16] E. Brézin and S. Hikami, Phys. Rev. E **55**, 4067 (1997).
[17] E. Brézin and S. Hikami, Phys. Rev. E **56**, 264 (1997).
[18] T. Guhr, Ann. Phys. (N.Y.) **250**, 145 (1996).